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Density Correlations in the Two-Dimensional Coulomb Gas

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We consider a two-dimensional Coulomb gas of positive and negative pointlike unit charges interacting via a logarithmic potential. The density (rather than the charge) correlation functions are studied. In the bulk, the form-factor theory of an equivalent sine-Gordon model is used to determine the density correlation length. At the surface of a rectilinear plain wall, the universality of the asymptotic behavior of the density correlations is suggested. A scaling analysis implies a local form of the compressibility sum rule near a hard wall. A symmetry of the Coulomb system with respect to the Möbius conformal transformation, which induces a gravitational source acting on the particle density, is established. Among the consequences, a universal term of the finite-size expansion of the grand potential is derived exactly for a disk geometry of the confining domain.

KEY WORDS: Coulomb systems; sine-Gordon model, correlations; surface properties; sum rules; universal finite-size correction.

1. INTRODUCTION AND SUMMARY

In this paper, we consider classical Coulomb systems in thermodynamic equilibrium. For the sake of simplicity, we will restrict ourselves to the case of a symmetric two-component plasma (TCP), i.e., a neutral system of two species of particles of opposite unit charges $\sigma = \pm 1$, living in a ν-dimensional space and interacting through the Coulomb interaction $\sigma_i \sigma_j v(|\mathbf{r}_i - \mathbf{r}_j|)$. In dimension ν, the Coulomb potential v at a spatial position $\mathbf{r} \in R^\nu$, induced by a unit charge at the origin, is the solution of the Poisson equation

$$\Delta v(\mathbf{r}) = -s_\nu \delta(\mathbf{r}) \tag{1.1}$$

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where s_ν is the surface area of the ν-dimensional unit sphere; $s_2 = 2\pi$ and $s_3 = 4\pi$. In particular,

$$v(\mathbf{r}) = -\ln(|\mathbf{r}|/r_0), \qquad \nu = 2 \tag{1.2a}$$

$$v(\mathbf{r}) = 1/|\mathbf{r}|, \qquad \nu = 3 \tag{1.2b}$$

where, for $\nu = 2$, the length scale r_0 will be set to unity without any loss of generality. For the case of pointlike particles, the singularity of $v(\mathbf{r})$ at the origin prevents the thermodynamic stability against the collapse of positive-negative pairs of charges: in two dimensions for small enough temperatures, in three dimensions for any temperature.

Let us now introduce some notations. We shall work in the grand canonical ensemble, characterized by the fixed domain D in which the TCP is confined, by the inverse temperature β, and by the couple of equal (constant) particle fugacities $z_+ = z_- = z$. The microscopic densities of charge and of the total particle number are defined respectively by

$$\hat{\rho}(\mathbf{r}) = \sum_\sigma \sigma \hat{n}_\sigma(\mathbf{r}), \qquad \hat{n}(\mathbf{r}) = \sum_\sigma \hat{n}_\sigma(\mathbf{r}) \tag{1.3}$$

where $\hat{n}_\sigma(\mathbf{r}) = \sum_i \delta_{\sigma,\sigma_i} \delta(\mathbf{r}-\mathbf{r}_i)$ is the microscopic density of particles of species $\sigma = \pm$ and i indexes the particles. The thermal average will be denoted by $\langle \cdots \rangle$. At one particle level, the total charge and particle number densities are given respectively by

$$\rho(\mathbf{r}) = \langle \hat{\rho}(\mathbf{r}) \rangle, \qquad n(\mathbf{r}) = \langle \hat{n}(\mathbf{r}) \rangle \tag{1.4}$$

Due to the charge $\pm$ symmetry, $n_+(\mathbf{r}) = n_-(\mathbf{r}) = n(\mathbf{r})/2$. At two-particle level, one introduces the two-body densities

$$\begin{aligned} n_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') &= \left\langle \sum_{i \neq j} \delta_{\sigma,\sigma_i} \delta_{\sigma',\sigma_j} \delta(\mathbf{r}-\mathbf{r}_i)\, \delta(\mathbf{r}'-\mathbf{r}_j) \right\rangle \\ &= \langle \hat{n}_\sigma(\mathbf{r})\, \hat{n}_{\sigma'}(\mathbf{r}') \rangle - \langle \hat{n}_\sigma(\mathbf{r}) \rangle\, \delta_{\sigma\sigma'} \delta(\mathbf{r}-\mathbf{r}') \end{aligned} \tag{1.5}$$

Clearly, $n_{++}(\mathbf{r}, \mathbf{r}') = n_{--}(\mathbf{r}, \mathbf{r}')$ and $n_{+-}(\mathbf{r}, \mathbf{r}') = n_{-+}(\mathbf{r}, \mathbf{r}')$. The corresponding Ursell functions read

$$U_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') = n_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') - n_\sigma(\mathbf{r})\, n_{\sigma'}(\mathbf{r}') \tag{1.6}$$

They will occur in the charge and density combinations

$$U_\rho(\mathbf{r}, \mathbf{r}') = \sum_{\sigma, \sigma'} \sigma\sigma' U_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \tag{1.7a}$$

$$U_n(\mathbf{r}, \mathbf{r}') = \sum_{\sigma, \sigma'} U_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \tag{1.7b}$$

respectively. The truncated charge-charge and density-density structure functions are defined by

$$S_\rho(\mathbf{r}, \mathbf{r}') = \langle \hat{\rho}(\mathbf{r})\, \hat{\rho}(\mathbf{r}') \rangle - \langle \hat{\rho}(\mathbf{r}) \rangle \langle \hat{\rho}(\mathbf{r}') \rangle \tag{1.8a}$$

$$S_n(\mathbf{r}, \mathbf{r}') = \langle \hat{n}(\mathbf{r})\, \hat{n}(\mathbf{r}') \rangle - \langle \hat{n}(\mathbf{r}) \rangle \langle \hat{n}(\mathbf{r}') \rangle \tag{1.8b}$$

respectively. It is useful to consider also the pair correlation functions

$$h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') = \frac{n_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}')}{n_\sigma(\mathbf{r})\, n_{\sigma'}(\mathbf{r}')} - 1 \tag{1.9}$$

in their charge and density combinations, defined by

$$h_\rho(\mathbf{r}, \mathbf{r}') = \tfrac{1}{4} \sum_{\sigma, \sigma'} \sigma\sigma' h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \tag{1.10a}$$

$$h_n(\mathbf{r}, \mathbf{r}') = \tfrac{1}{4} \sum_{\sigma, \sigma'} h_{\sigma\sigma'}(\mathbf{r}, \mathbf{r}') \tag{1.10b}$$

As any Coulomb system, the TCP admits a description in the Debye–Hückel (high-temperature) limit.[1]

In two dimensions and for the case of pointlike charged particles, the particle density n is an "irrelevant" variable which only scales the distance. The complete description of the bulk thermodynamics is available in the whole range of inverse temperatures $\beta < 2$ where the plasma is stable against the collapse of positive-negative pairs of charges. The exact equation of state

$$\beta p = n(1 - \beta/4) \tag{1.11}$$

where p is the pressure and n the total particle density, has been known for a very long time.[2, 3] The evaluation of other thermodynamic quantities (free energy, internal energy, specific heat, etc.) can be based on an explicit density-fugacity $(n - z)$ relationship. This relationship was obtained only recently[4] via a mapping onto a classical two-dimensional sine-Gordon theory. Surface thermodynamics (surface tension) of the two-dimensional TCP in contact with a rectilinear dielectric wall was derived for specific boundary conditions in refs. 5 and 6. The large-distance behavior of the bulk charge-charge correlation function was given in ref. 7 by exploring the form-factor method for the equivalent sine-Gordon theory. Exact formulae for pair correlations for an arbitrary interparticle distance are available just at the collapse point $\beta = 2$,[8, 9] which corresponds to the free-fermion point of an equivalent Thirring model.

The long-range tail of the Coulomb force causes screening, and thus give rise to exact constraints for the charge correlations for any value of β in the conducting regime (for a review, see ref. 10).

In the bulk, the charge structure function $S_\rho(\mathbf{r}, \mathbf{r}') = S_\rho(|\mathbf{r}-\mathbf{r}'|)$ obeys the Stillinger–Lovett sum rules,[11] namely the zeroth-moment condition

$$\int \mathrm{d}\mathbf{r}\; S_\rho(\mathbf{r}) = 0 \tag{1.12}$$

and the second-moment condition

$$\int \mathrm{d}\mathbf{r}\; |\mathbf{r}|^2\, S_\rho(\mathbf{r}) = -\frac{\nu}{\beta\pi(\nu-1)}, \qquad \nu = 2, 3 \tag{1.13}$$

Hereinafter, without any loss of generality, we put the dielectric constant ϵ of a medium in which the plasma lives equal to unity.

Let us now introduce a semi-infinite TCP which occupies the half space $x > 0$, and denote by $\mathbf{y}$ the set of $(\nu-1)$ coordinates normal to x. The plane at $x = 0$ is a hard wall impenetrable to the particles. The half-space $x < 0$ is assumed to be filled with a material of dielectric constant ϵ_W: a particle of unit charge at the point $\mathbf{r} = (x > 0, \mathbf{y})$ has an electric image of charge $(1-\epsilon_W)/(1+\epsilon_W)$ at the point $\mathbf{r}^* = (-x, \mathbf{y})$.[12] Due to invariance with respect to translations along the wall and rotations around the x direction,

$$S_\rho(\mathbf{r}, \mathbf{r}') = S_\rho(x, x'; |\mathbf{y}-\mathbf{y}'|) = S_\rho(x', x; |\mathbf{y}-\mathbf{y}'|) \tag{1.14}$$

The electroneutrality condition (1.12) takes the form

$$\int_0^\infty \mathrm{d}x' \int \mathrm{d}\mathbf{y}\; S_\rho(x, x'; \mathbf{y}) = 0 \tag{1.15}$$

The Carnie and Chan[13] generalization of the second-moment condition (1.13) results in the dipole sum rule[14, 15]

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}x' \int \mathrm{d}\mathbf{y}\; (x'-x)\, S_\rho(x, x'; \mathbf{y}) = -\frac{1}{2\beta\pi(\nu-1)}, \qquad \nu = 2, 3 \tag{1.16}$$

The asymmetry of the screening cloud of a charged particle sitting near the wall induces a long-range tail in the charge correlation along the wall,[16, 17] except of the cases $\epsilon_W = 0$ (ideal dielectric wall) and $\epsilon_W = \infty$ (ideal conductor wall) which will be excluded from our considerations. One expects an asymptotic power-law decay

$$S_\rho(x, x'; \mathbf{y}) \sim \frac{f_\rho(x, x')}{|\mathbf{y}|^\nu}, \qquad |\mathbf{y}| \to \infty \tag{1.17}$$

where $f_\rho(x, x')$ obeys the sum rule[18, 19]

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}x' \, f_\rho(x, x') = -\frac{\epsilon_W}{2\beta[\pi(\nu-1)]^2}, \qquad \nu = 2, 3 \tag{1.18}$$

Recently,[20] it was proven that for any value of $x \geqslant 0$ it holds

$$\int_0^\infty \mathrm{d}x' \int \mathrm{d}\mathbf{y} \, (x'-x) \, S_\rho(x, x'; \mathbf{y}) = \frac{1}{\epsilon_W} \pi(\nu-1) \int \mathrm{d}x' \, f_\rho(x, x'), \qquad \nu = 2, 3 \tag{1.19}$$

When both sides of (1.19) are integrated over x from 0 to ∞, it is clear that the sum rule (1.18) for f_ρ is a direct consequence of the dipole sum rule (1.16), and vice versa.

As concerns the density correlation function, according to the general theory of fluids,[21] the zeroth moment of the bulk density structure function $S_n(\mathbf{r}, \mathbf{r}') = S_n(|\mathbf{r}-\mathbf{r}'|)$ is related to the isothermal compressibility

$$\chi_\beta = \frac{1}{n} \left(\frac{\partial n}{\partial p} \right)_\beta \tag{1.20}$$

via

$$\frac{1}{n^2} \int \mathrm{d}\mathbf{r} \, S_n(\mathbf{r}) = \frac{1}{\beta} \chi_\beta \tag{1.21}$$

In two dimensions and for the case of pointlike charged particles, with the use of the exact equation of state (1.11), one gets explicitly[22]

$$\int \mathrm{d}^2 r \, S_n(r) = \frac{n}{1-(\beta/4)} \tag{1.22}$$

Recently, using the technique of a renormalized Mayer expansion,[23] and in particular a remarkable "cancellation property" of specific families of renormalized diagrams,[24] the second moment of S_n was shown to have a simple value[25]

$$\int \mathrm{d}^2 r\, |\mathbf{r}|^2\, S_n(r) = \frac{1}{12\pi(1-(\beta/4))^2} \tag{1.23}$$

The universal finite-size properties of two-dimensional critical systems with *short-range* interactions among constituents are well understood within the principle of conformal invariance.[26–30] For a finite system of characteristic size R, at a critical point, the dimensionless grand potential $\beta\Omega = -\ln \Xi$ (Ξ is the grand partition function) has a large-R expansion of the form

$$\beta\Omega = AR^2 + BR - \frac{c\chi}{6} \ln R + \text{const} + \cdots \tag{1.24}$$

The coefficients A and B of the bulk and surface parts are non-universal. The coefficient of the $\ln R$-term is universal, dependent only on the conformal anomaly number c of the critical theory and on the Euler number χ of the manifold on which the system is confined. In general, $\chi = 2-2h-b$, where h is the number of handles and b the number of boundaries of the manifold ($\chi = 2$ for a sphere, $\chi = 1$ for a disk, $\chi = 0$ for an anulus or a torus). The grand potential of the two-dimensional TCP is supposed to exhibit a universal finite-size correction of type (1.24) at any temperature of the conducting regime. Plausible, but not always rigorously justified, arguments for a critical-like behavior were first given for Coulomb gases with periodic boundary conditions,[31] then for Coulomb systems confined to a domain by plain hard walls,[32] by ideal-conductor walls[33] and finally by ideal-dielectric boundaries.[34, 35] The explicit checks were done at the exactly solvable $\beta = 2$ inverse temperature for various geometries of confining domains. Only very recently,[36] a direct derivation of the universal finite-size correction term was done for the specific case of the TCP living on the surface of a sphere of radius R. By combining the method of stereographic projection of the sphere onto an infinite plane with the linear response theory, the prefactor to the universal $\ln R$ correction term was related to the bulk second moment of the density structure function S_n, Eq. (1.23). The obtained result confirms the prediction (1.24) for a Coulomb system, as if we had $c = -1$, in full agreement with heuristic approaches and exact results at $\beta = 2$.[31–35]

In general, due to screening phenomena, the sum rules for the charge structure function S_ρ are not modified by a short-distance regularization of the Coulomb potential. On the other hand, the moments of the density structure function S_n *depend* on the particular form of the short-range particle interactions. For the case of two dimensions and the pointlike character of charged particles on which we shall concentrate in this work, the zeroth-moment (1.12) and the second-moment (1.13) Stillinger–Lovett conditions for S_ρ have their exact counterparts (1.22) and (1.23), respectively, for S_n. Instead of screening, the scaling properties and the critical-like state are relevant. The aim of this paper is to document how these features of the two-dimensional Coulomb fluids manifest themselves in the density correlation functions (in the bulk or close to a boundary) and in the universal finite-size correction term of the grand potential.

Section 2 is devoted to the density correlations in the bulk. The Debye–Hückel $\beta \to 0$ limit is presented in Section 2.1, the exactly solvable $\beta = 2$ case is treated in Section 2.2. A general analysis, based on the form-factor theory of the equivalent sine-Gordon model in analogy with ref. 7, is given in Section 2.3. For comparison, the corresponding formulae for the charge correlations are presented, too.

Density correlations near a rectilinear hard wall are analysed in Section 3. In Section 3.1, we present the results of the Debye–Hückel $\beta \to 0$ limit. Section 3.2 deals with the exactly solvable $\beta = 2$ case for the plain hard wall with $\epsilon_W = 1$. In comparison with formula (1.17) for S_ρ taken at dimension $\nu = 2$, a more rapid, but still power-law, asymptotic decay of S_n along the boundary is observed,

$$S_n(x, x'; y) \sim \frac{f_n(x, x')}{y^4} \tag{1.25}$$

In Section 3.3, at least for the considered plain hard wall, the universal form of the function $f_n(x, x')$ at the boundary $x = x' = 0$ is conjectured,

$$f_n(0, 0) = \frac{1}{2\pi^2} \tag{1.26}$$

Here, we also suggest explicit forms of $f_\rho(x, x')$ and $f_n(x, x')$ for any value of β.

In Section 4, we explore the scaling properties of the two-dimensional TCP confined to a disk of radius R, for the sake of simplicity by an uncharged plain hard wall. We present some important formulae which are used in the subsequent sections. The rectilinear wall is obtained as the limiting $R \to \infty$ case of the disk. As a by-product of the formalism, we derive a local form of the compressibility (zeroth-moment) sum rule (1.22) for S_n near the rectilinear hard wall.

In Section 5, by using a specific (namely Möbius) conformal transformation of particle coordinates we show how the two-dimensional TCP can be mapped onto the one under the action of a gravitational source, acting in the same way on both positively and negatively charged particles. The mapping implies a new sum rule for S_n in the disk geometry. In this section, the new sum rule is used to derive the counterpart of the bulk second-moment sum rule for S_n (1.23) near a rectilinear wall.

Section 6 deals with the universal finite-size correction $\ln R$-term for the disk geometry. Using the new sum rule for S_n derived in Section 5, the universal term is confirmed at any value of β. This term has already been calculated at $\beta = 2$ in ref. 32.

A brief recapitulation and some concluding remarks are given in Section 7.

2. BULK DENSITY CORRELATIONS

2.1. Weak Coupling

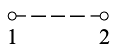
In this subsection, we derive the asymptotic form of the density correlation function $h_n(r)$ in the bulk, at the lowest order in β. It should be noted that β-expansions of correlation functions must be taken for a fixed value of the inverse Debye length $\kappa = (2\pi\beta n)^{1/2}$, since κ only fixes the length scale.

In the present case of a charge-symmetrical two-component plasma, the ordinary Ornstein-Zernike (OZ) equation split into two independent relations for the charge-charge and density-density functions[25]

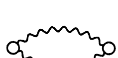
$$h_\rho = c_\rho + c_\rho * n * h_\rho \tag{2.1a}$$

$$h_n = c_n + c_n * n * h_n \tag{2.1b}$$

respectively, where $*$ denotes a convolution product. In the renormalized Mayer expansion,[4, 25] at the lowest order in β, the charge direct correlation function is given by

$c_\rho(1, 2) =$  $= -\beta v(1, 2)$ (2.2)

with v being the Coulomb potential, and the density direct correlation function is given by the Meeron diagram

$c_n(1, 2) =$  $= \frac{1}{2!} K^2(1, 2)$ (2.3)

The wavy line denotes the renormalized bond K (sum of chains). In the bulk, $K(r) = -\beta K_0(\kappa r)$, where K_0 is a modified Bessel function.

Inserting (2.2) into (2.1a), one gets the definition of K. This is why

$$h_\rho(r) = -\beta K_0(\kappa r) \sim -\beta \left(\frac{\pi}{2\kappa r}\right)^{1/2} \exp(-\kappa r) \tag{2.4}$$

since $K_0(x)$ has the asymptotic form $[\pi/(2x)]^{1/2} \exp(-x)$. In the OZ relation (2.1b), where c_n (2.3) is of order β^2, the convolution term is easily seen to be of higher order β^3. Thus, at lowest order in β, $h_n(r) = c_n(r)$,

$$h_n(r) = \frac{\beta^2}{2} K_0^2(\kappa r) \sim \frac{\pi \beta^2}{4\kappa r} \exp(-2\kappa r) \tag{2.5}$$

The same analysis can be done in dimension $\nu = 3$, where

$$K(r) = -\frac{\beta}{r} \exp(-\kappa r) \tag{2.6}$$

where now $\kappa = (4\pi\beta n)^{1/2}$.[37] In this case, the large-r behavior is not touched by an inevitable short-distance regularization of the Coulomb potential.

2.2. $\beta = 2$

When $\beta = 2$, for a fixed fugacity z, the particle density $n \to \infty$ and $\{h_\rho, h_n\} \to 0$. However, the Ursell functions $U_\rho = n^2 h_\rho$ and $U_n = n^2 h_n$ are exactly known[8, 9] as

$$U_\rho(r) = -2 \left(\frac{m^2}{2\pi}\right)^2 [K_1^2(mr) + K_0^2(mr)] \tag{2.7}$$

and

$$U_n(r) = 2 \left(\frac{m^2}{2\pi}\right)^2 [K_1^2(mr) - K_0^2(mr)] \tag{2.8}$$

where $m = 2\pi z$ (z is the fugacity). Replacing the modified Bessel functions by their asymptotic expansions

$$K_0(x) = \left(\frac{\pi}{2x}\right)^{1/2} \mathrm{e}^{-x} \left[1 - \frac{1}{8x} + \cdots\right] \tag{2.9a}$$

$$K_1(x) = \left(\frac{\pi}{2x}\right)^{1/2} \mathrm{e}^{-x} \left[1 + \frac{3}{8x} + \cdots\right] \tag{2.9b}$$

gives the asymptotic form of $U_\rho(r)$ as

$$U_\rho(r) \sim -\frac{m^3}{2\pi r} \exp(-2mr) \tag{2.10}$$

and the asymptotic form of $U_n(r)$ as

$$U_n(r) \sim \frac{m^2}{4\pi r^2} \exp(-2mr) \tag{2.11}$$

2.3. Form-Factor Analysis

We now determine the large-distance behavior of $h_n(r)$ in the whole stability interval of inverse temperatures $0 < \beta < 2$ via a form-factor analysis.

The grand partition function of the TCP can be turned into (see, e.g., ref. 38)

$$\Xi = \frac{\int \mathscr{D}\phi \exp(-S(z))}{\int \mathscr{D}\phi \exp(-S(0))} \tag{2.12}$$

where

$$S(z) = \int \mathrm{d}^2 r \left[\frac{1}{16\pi} (\nabla\phi)^2 - 2z \cos(b\phi) \right] \tag{2.13a}$$

$$b^2 = \beta/4 \tag{2.13b}$$

is the Euclidean action of the classical sine-Gordon model. In the sine-Gordon representation, the density of particles of one sign $\sigma = \pm$ is

$$n_\sigma = z_\sigma \langle \mathrm{e}^{\mathrm{i}\sigma b\phi} \rangle \tag{2.14}$$

where $\langle \cdots \rangle$ denotes the averaging over the sine-Gordon action (2.13), two-body densities (1.5) are expressible as follows

$$n_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') = z_\sigma z_{\sigma'} \langle \mathrm{e}^{\mathrm{i}\sigma b\phi(\mathbf{r})} \mathrm{e}^{\mathrm{i}\sigma' b\phi(\mathbf{r}')} \rangle \tag{2.15}$$

etc. The parameter $z = z_+ = z_-$, which is the fugacity renormalized by a (diverging) self-energy term, gets a precise meaning when one fixes the normalization of the cos-field. For the TCP,[4] this normalization is given by the short-distance behavior

$$n_{+-}(\mathbf{r},\mathbf{r}') \sim z_+ z_- \, |\mathbf{r}-\mathbf{r}'|^{-\beta} \qquad \text{as} \quad |\mathbf{r}-\mathbf{r}'| \to 0 \tag{2.16}$$

dominated by the Boltzmann factor of the Coulomb potential. The corresponding formula in the sine-Gordon picture

$$\langle e^{ib\phi(\mathbf{r})} e^{-ib\phi(\mathbf{r}')} \rangle \sim |\mathbf{r}-\mathbf{r}'|^{-4b^2} \qquad \text{as} \quad |\mathbf{r}-\mathbf{r}'| \to 0 \tag{2.17}$$

is known in quantum field theory as the conformal normalization.

The sine-Gordon model (2.13) is massive in the region $0 < b^2 < 1$ ($0 < \beta < 4$). It is integrable:[39] its particle spectrum consists of one soliton-antisoliton pair of equal masses M and of soliton-antisoliton bound states ("breathers") $\{B_j;\ j = 1, 2, \ldots < 1/\xi\}$. Their number depends on the inverse of the parameter

$$\xi = \frac{b^2}{1-b^2} \qquad \left(= \frac{\beta}{4-\beta} \right) \tag{2.18}$$

The mass of the B_j-breather is given by

$$m_j = 2M \sin\left(\frac{\pi\xi}{2}\, j\right) \tag{2.19}$$

and the breather disappears from the spectrum just when $m_j = 2M$. Under the conformal normalization (2.17), the relationship between the soliton mass M and the parameter z was established in ref. 40,

$$z = \frac{\Gamma(b^2)}{\pi \Gamma(1-b^2)} \left[M \frac{\sqrt{\pi}\, \Gamma((1+\xi)/2)}{2\Gamma(\xi/2)} \right]^{2-2b^2} \tag{2.20}$$

where Γ stands for the Gamma function. Using the Thermodynamic Bethe ansatz, the specific quantity

$$\lim_{V \to \infty} \frac{1}{V} \ln \Xi = \frac{m_1^2}{8 \sin(\pi\xi)} \tag{2.21}$$

was found in ref. 41. As a thermodynamic result,[4]

$$n = \frac{M^2}{4(1-b^2)} \tan\left(\frac{\pi\xi}{2}\right) \tag{2.22}$$

Note that as β approaches the collapse value 2, for a fixed z, M is finite and $n \to \infty$ as it should be.

For the underlying sine-Gordon theory, the two-point truncated correlation functions of local operators $\mathcal{O}_a$ (a is a free parameter) can be formally written as an infinite convergent series over multi-particle intermediate states,

$$\langle \mathcal{O}_a(\mathbf{r})\, \mathcal{O}_{a'}(\mathbf{r}')\rangle_{\mathrm{T}} = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\epsilon_1,\ldots,\epsilon_N} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N}{(2\pi)^N} F_a(\theta_1,\ldots,\theta_N)_{\epsilon_1\cdots\epsilon_N} \\ \times\, {}^{\epsilon_N\cdots\epsilon_1}F_{a'}(\theta_N,\ldots,\theta_1) \exp\left(-|\mathbf{r}-\mathbf{r}'| \sum_{j=1}^{N} m_{\epsilon_j} \cosh\theta_j \right) \tag{2.23}$$

where ϵ indexes the particles [$\epsilon = +(-)$ for a soliton (antisoliton) and $\epsilon = j$ for a breather B_j ($j = 1, 2,\ldots$)] and θ is the particle rapidity. The form factors

$$F_a(\theta_1,\ldots,\theta_N)_{\epsilon_1\cdots\epsilon_N} = \langle 0|\, \mathcal{O}_a(\mathbf{0})\, |Z_{\epsilon_1}(\theta_1),\ldots, Z_{\epsilon_N}(\theta_N)\rangle \tag{2.24a}$$

$${}^{\epsilon_N\cdots\epsilon_1}F_{a'}(\theta_N,\ldots,\theta_1) = \langle Z_{\epsilon_N}(\theta_N),\ldots, Z_{\epsilon_1}(\theta_1)|\, \mathcal{O}_{a'}(\mathbf{0})\, |0\rangle \tag{2.24b}$$

are the matrix elements of the operator at the origin, between an N-particle in-state (as a linear superposition of free one-particle states $|Z_\epsilon(\theta)\rangle$) and the vacuum.

In the limit $|\mathbf{r}-\mathbf{r}'| \to \infty$, the dominant contribution to the truncated correlation function in (2.23) comes from a multi-particle state with the minimum value of the total particle mass $\sum_{j=1}^{N} m_{\epsilon_j}$, at the point of vanishing rapidities $\theta_j \to 0$. Due to topological reasons, solitons and antisolitons coexist in pairs, the total mass of the pair being $2M$. The breathers B_j with masses given by relation (2.19) are lighter and therefore, when they exist and their form-factor contributions do not vanish, they are the best candidates for governing the asymptotic behavior of the two-point correlation function.

The form factors of an exponential operator $\mathcal{O}_a = \exp(\mathrm{i}a\phi(\mathbf{r}))$ for various combinations of particles were calculated in refs. 42–44. The one-breather form factors, which do not depend on the rapidity, posses the following general structure

$$\langle 0|\, \mathrm{e}^{\mathrm{i}\sigma b\phi}\, |B_j(\theta)\rangle = \langle B_j(\theta)|\, \mathrm{e}^{\mathrm{i}\sigma b\phi}\, |0\rangle \propto \sigma^j \langle \mathrm{e}^{\mathrm{i}\sigma b\phi}\rangle \sin\left(\frac{\pi j}{2}\right) \tag{2.25}$$

where the full dependence on $\sigma = \pm 1$ is presented. These form factors are nonzero only if j = odd integer. Due to the invariance of the sine-Gordon action (2.13) with respect to the transformation $\phi \to -\phi$, it holds $\langle e^{ib\phi} \rangle = \langle e^{-ib\phi} \rangle$. With regard to (2.15), the total contribution of a given breather B_j (j odd) to the charge h_ρ and density h_n correlation functions (1.10) is proportional to

$$h_\rho \propto \sum_{\sigma, \sigma' = \pm 1} \sigma\sigma' \langle 0 | \, e^{i\sigma b\phi} \, | B_j(\theta) \rangle \langle B_j(\theta) | \, e^{i\sigma' b\phi} \, | 0 \rangle \tag{2.26a}$$

$$h_n \propto \sum_{\sigma, \sigma' = \pm 1} \langle 0 | \, e^{i\sigma b\phi} \, | B_j(\theta) \rangle \langle B_j(\theta) | \, e^{i\sigma' b\phi} \, | 0 \rangle \tag{2.26b}$$

Inserting (2.25) into (2.26) one observes that one-breather states contribute only to h_ρ. Using (2.22) in (2.19), the mass of the lightest (elementary) B_1-breather is

$$m_1 = \kappa \left[\frac{\sin(\pi\beta/(4-\beta))}{\pi\beta/(4-\beta)} \right]^{1/2} \tag{2.27}$$

where κ denotes as usually the inverse Debye length. At asymptotically large r, (2.23) gives

$$h_\rho(r) \propto \exp(-m_1 r), \qquad 0 < \beta < 2 \tag{2.28}$$

in agreement with the $\beta \to 0$ limit (2.4). At the free-fermion point $\beta = 2$, the B_1-breather disappears, and the soliton-antisoliton pair with mass $2M$ determines the correlation length:

$$U_\rho(r) \propto \exp(-2Mr), \qquad \beta = 2 \tag{2.29}$$

From (2.20), $M = 2\pi z = m$ at $\beta = 2$ ($b^2 = 1/2$), and the asymptotic form (2.10) is reproduced. Note that from (2.19) $m_1 \to 2M$ as $\beta \to 2$, and so the inverse correlation length varies continuously near $\beta = 2$. The explicit inverse-power law dependence of prefactors of asymptotic formulae (2.28) and (2.29) was presented in ref. 7.

At small β, the large-distance behavior of h_n is determined by the two-B_1-breather state. Indeed, the corresponding form factor[7, 43]

$$\langle 0 | \, e^{i\sigma b\phi} \, | B_1(\theta_2), B_1(\theta_1) \rangle = \langle B_1(\theta_2), B_1(\theta_1) | \, e^{i\sigma b\phi} \, | 0 \rangle \propto \sigma^2 \langle e^{i\sigma b\phi} \rangle \tag{2.30}$$

(where the full dependence on $\sigma = \pm 1$ is given) has the necessary $\sigma \to -\sigma$ symmetry and thus contributes to h_n. It follows from (2.19) that the mass of two B_1-breathers, $2m_1$, is smaller than the one of the soliton-antisoliton pair, $2M$, in the region $\beta < 1$. Consequently,

$$h_n(r) \propto \exp(-2m_1 r), \qquad 0 < \beta < 1 \tag{2.31}$$

at large distance r, in agreement with the $\beta \to 0$ limit (2.5). It stands to reason that subsequently the soliton-antisoliton pair determines the large-distance asymptotic of h_n,

$$h_n(r) \propto \exp(-2Mr), \qquad 1 \leqslant \beta \leqslant 2 \tag{2.32}$$

where the β-dependence of M can be deduced from Eq. (2.22). At $\beta = 2$, the result (2.11) is recovered. We omit a tedious calculation of the inverse-power law dependence of prefactors on distance in (2.31) and (2.32).

One concludes that, for a given $\beta < 2$, the large-distance exponential decay of h_n is faster than the one of h_ρ. Although the correlation lengths depend continuously on β for both h_ρ and h_n, the derivative of the density correlation length with respect to β is discontinuous at $\beta = 1$. The correlation lengths coincide just at the collapse $\beta = 2$ point, where, as is clear from the exact formulae (2.10) and (2.11), h_ρ and h_n differ from one another only by inverse-power law prefactors.

3. DENSITY CORRELATIONS NEAR A RECTILINEAR HARD WALL

3.1. Weak Coupling

Like in Section 2.1, at the lowest order in β, the direct correlation functions c_ρ and c_n are given by relations (2.2) and (2.3), respectively. However, now, the wavy line K, the sum of the chain diagrams, is a function $K(x, x'; |y-y'|)$ which must be calculated with the wall taken into account. An arbitrary ϵ_W value can be assumed for the dielectric constant of the wall material. At the lowest order in β, K can be computed with the density $n(x)$ replaced by its bulk value n. This K has been implicitly considered in refs. 17 and 20, where it was used as the charge correlation function, $h_\rho = K$. Except in the cases $\epsilon_W = \infty$ (ideal conductor wall) and $\epsilon_W = 0$ (ideal dielectric wall), K has an algebraic decay, for large $|y-y'|$, of the form $-\epsilon_W/[\pi n(y-y')^2] \exp[-\kappa(x+x')]$. Therefore,

$$U_\rho(x, x'; |y-y'|) = n^2 K(x, x'; |y-y'|) \sim -\frac{\epsilon_W n}{\pi(y-y')^2} \exp[-\kappa(x+x')] \tag{3.1}$$

The Meeron diagram (2.3) gives for the asymptotic form of the density Ursell function

$$U_n(x, x'; |y-y'|) = n^2 c_n(x, x'; |y-y'|) \sim \frac{\epsilon_W^2}{2\pi^2 (y-y')^4} \exp[-2\kappa(x+x')] \tag{3.2}$$

These formulae can be generalized to $\nu = 2, 3$ dimensions:

$$U_\rho(x, x'; |\mathbf{y}|) \sim -\frac{\epsilon_W n}{(\nu-1)\,\pi\,|\mathbf{y}|^\nu} \exp[-\kappa(x+x')] \tag{3.3}$$

$$U_n(x, x'; |\mathbf{y}|) \sim \frac{\epsilon_W^2}{2(\nu-1)^2\,\pi^2\,|\mathbf{y}|^{2\nu}} \exp[-2\kappa(x+x')] \tag{3.4}$$

with $\kappa = [2(\nu-1)\,\pi\beta n]^{1/2}$.

3.2. $\beta = 2$

When $\beta = 2$ and $\epsilon_W = 1$, the charge and density Ursell functions can be expressed in terms of auxiliary functions $g_{\sigma+}$, where $\sigma = \pm$, as[9]

$$U_\rho(x, x'; |y-y'|) = -2m^2[|g_{-+}(x, x'; y-y')|^2 + |g_{++}(x, x'; y-y')|^2] \tag{3.5a}$$

$$U_n(x, x'; |y-y'|) = 2m^2[|g_{-+}(x, x'; y-y')|^2 - |g_{++}(x, x'; y-y')|^2] \tag{3.5b}$$

where $m = 2\pi z$ (z is the fugacity). It will now be shown that, near the wall, each g function has a slow algebraic decay for large $|y-y'|$.

Each g is a sum of two terms

$$g_{\sigma+}(x, x'; y-y') = g_{\sigma+}^{\text{bulk}}(\mathbf{r}-\mathbf{r}') + g_{\sigma+}^{\text{wall}}(x, x'; y-y') \tag{3.6}$$

The first term in the rhs of (3.6) is the same as in the bulk. It has a fast (exponential) decay and does not contribute to the asymptotic form, which is entirely due to the second term. This second term is determined by its Fourier transform as

$$g_{\sigma+}^{\text{wall}}(x, x'; y-y') = \int_{-\infty}^{\infty} \frac{dl}{2\pi}\, \tilde{g}_{\sigma+}^{\text{wall}}(x, x'; l) \exp[il(y-y')] \tag{3.7}$$

These Fourier transforms are

$$\tilde{g}^{\text{wall}}_{++}(x, x'; l) = -\frac{m}{2k} \exp[-k(x+x')], \qquad l<0 \tag{3.8a}$$

$$\tilde{g}^{\text{wall}}_{++}(x, x'; l) = \frac{m(k-l)}{2k(k+l)} \exp[-k(x+x')], \qquad l>0 \tag{3.8b}$$

$$\tilde{g}^{\text{wall}}_{-+}(x, x'; l) = -\frac{k+l}{2k} \exp[-k(x+x')], \qquad l<0 \tag{3.8c}$$

$$\tilde{g}^{\text{wall}}_{-+}(x, x'; l) = \frac{k-l}{2k} \exp[-k(x+x')], \qquad l>0 \tag{3.8d}$$

where $k=(m^2+l^2)^{1/2}$. Thus, $\tilde{g}^{\text{wall}}_{\sigma+}(x, x'; l)$ is a function of l singular at $l=0$. This singularity generates in $g^{\text{wall}}_{\sigma+}(x, x'; y-y')$ an algebraic decay at large $|y-y'|$. The corresponding asymptotic expansion is obtained by splitting the integral in (3.7) into the $l<0$ and $l>0$ contributions, and evaluating each contribution by successive integration by parts. For instance, calling for simplicity $N(l)$ the function $\tilde{g}^{\text{wall}}_{\sigma+}(x, x'; l)$ when $l<0$, one obtains the asymptotic expansion

$$\int_{-\infty}^{0} \frac{\mathrm{d}l}{2\pi} N(l) \exp[\mathrm{i}l(y-y')]$$
$$= \frac{1}{2\pi}\left\{ -\frac{\mathrm{i}}{y-y'} N(0) + \frac{1}{(y-y')^2} N'(0) + \frac{\mathrm{i}}{(y-y')^3} N''(0) + \cdots \right\} \tag{3.9}$$

A similar expansion holds for $P(l)$, the function $\tilde{g}^{\text{wall}}_{\sigma+}(x, x'; l)$ when $l>0$. Using these expansions finally gives

$$|g_{++}(x, x'; y-y')|^2$$
$$= \frac{1}{4\pi^2}\left\{ \frac{1}{(y-y')^2} + \left[-\frac{1}{m^2} + \frac{2(x+x')}{m} \right] \frac{1}{(y-y')^4} + \cdots \right\} \exp[-2m(x+x')] \tag{3.10a}$$

and

$$|g_{-+}(x, x'; y-y')|^2 = \frac{1}{4\pi^2}\left\{ \frac{1}{(y-y')^2} + \frac{2(x+x')}{m(y-y')^4} + \cdots \right\} \exp[-2m(x+x')] \tag{3.10b}$$

The dominant term in Eqs. (3.10), of order $1/(y-y')^2$, determines the asymptotic form of the charge Ursell function (3.5a),[20]

$$U_\rho(x, x'; |y-y'|) \sim -\left(\frac{m}{\pi}\right)^2 \frac{1}{(y-y')^2} \exp[-2m(x+x')] \tag{3.11}$$

But this dominant term cancels out in the expression (3.5b) of the density Ursell function, the asymptotic form of which is governed by the subdominant term in Eqs. (3.10), of order $1/(y-y')^4$. The final result is

$$U_n(x, x'; |y-y'|) \sim \frac{1}{2\pi^2(y-y')^4} \exp[-2m(x+x')] \tag{3.12}$$

3.3. Conjectures

The functions $U_{\rho,n}(r)$ and $S_{\rho,n}(r)$ differ from one another only by a term containing $\delta(r)$, which has no effect on their identical large-distance behavior. With regard to the definitions of the asymptotic characteristics f_ρ (1.17) and f_n (1.25), the results of the two previous subsections can be summarized, for $v=2$ dimensions and $\epsilon_W=1$, by

$$f_\rho(x, x') = -\frac{n}{\pi} \exp[-\kappa(x+x')], \qquad \beta \to 0 \tag{3.13a}$$

$$f_\rho(x, x') = -\left(\frac{m}{\pi}\right)^2 \exp[-2m(x+x')], \qquad \beta = 2 \tag{3.13b}$$

and

$$f_n(x, x') = \frac{1}{2\pi^2} \exp[-2\kappa(x+x')], \qquad \beta \to 0 \tag{3.14a}$$

$$f_n(x, x') = \frac{1}{2\pi^2} \exp[-2m(x+x')], \qquad \beta = 2 \tag{3.14b}$$

The explicit forms of f_ρ and f_n in the two exactly solvable cases have an appealing feature: both charge and density functions factorize in x and x' particle coordinates. This can be intuitively explained by the fact that one studies the leading asymptotic $y \to \infty$ limit of pair correlations along the wall, in which probably there is no correlation between x and x' coordinates of particles. In both $\beta \to 0$ and $\beta = 2$ cases, the decay of factorized functions into the bulk has the exponential form obtained for the corresponding bulk functions in Sections 2.1 and 2.2. With regard to the general form-factor analysis in Section 2.3, it is therefore tempting to write down

$$f_\rho(x, x') = -\frac{m_1^2}{2\beta\pi^2} \exp[-m_1(x+x')], \qquad 0 < \beta \leqslant 2 \tag{3.15}$$

with the mass m_1 of the elementary B_1-breather given by formula (2.27). This formula reproduces correctly the two solvable cases (3.13) and reflects the dominance of the lightest particle in the sine-Gordon spectrum in the large-distance behavior of charge correlation functions. The value of the prefactor in (3.15) is fixed by the sum rule (1.18) with $\nu = 2$ and $\epsilon_W = 1$. As concerns the density asymptotic characteristic $f_n(x, x')$ (3.14), the prefactor of the exponential acquires the same value in the $\beta \to 0$ limit as well as at $\beta = 2$, what motivates us to suggest, taking into account large-distance asymptotics (2.31) and (2.32), that

$$f_n(x, x') = \frac{1}{2\pi^2} \exp[-2m_1(x+x')], \qquad 0 < \beta < 1 \tag{3.16a}$$

$$f_n(x, x') = \frac{1}{2\pi^2} \exp[-2M(x+x')], \qquad 1 < \beta \leqslant 2 \tag{3.16b}$$

In particular, at the boundary $x = x' = 0$, $f_n(0, 0)$ is supposed to have a universal value $1/(2\pi^2)$ independent of β. We were not able to give some general argument for such a result.

We emphasize that formulae (3.15) and (3.16) are only conjectures which must be verified, for example, within a systematic weak-coupling expansion in the presence of a dielectric wall. Such calculations for the correlation functions are tedious and far from simple, and will not be presented here.

4. SCALING ANALYSIS IN A DISK

Under the neutrality constraint $N_+ = N_-$, the grand partition function of the two-dimensional symmetric TCP in a domain D, bounded by an impermeable hard wall (for simplicity, uncharged and with no image forces, $\epsilon_W = 1$) is written as

$$\Xi = \sum_{N=0}^{\infty} \frac{z^{2N}}{(N!)^2} Q_{N,N} \tag{4.1a}$$

$$Q_{N,N} = \int_D \prod_{i=1}^{N} \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \, W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \tag{4.1b}$$

where $Q_{N,N}$ is the configuration integral and

$$W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) = \frac{\prod_{(i<j)=1}^{N} |\mathbf{p}_i - \mathbf{p}_j|^\beta \, |\mathbf{n}_i - \mathbf{n}_j|^\beta}{\prod_{i,j=1}^{N} |\mathbf{p}_i - \mathbf{n}_j|^\beta} \tag{4.2}$$

denotes the interaction Boltzmann weight of N particles of charge $+1$ with coordinates $\{\mathbf{p}_i\}_{i=1}^N$ and N particles of charge -1 with coordinates $\{\mathbf{n}_i\}_{i=1}^N$. From the explicit representations

$$\langle \hat{n}(\mathbf{r}) \rangle = \frac{1}{\Xi} \sum_{N=0}^{\infty} \frac{z^{2N}}{(N!)^2} \int_D \prod_{i=1}^{N} \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \, W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \times \sum_{i=1}^{N} [\delta(\mathbf{r} - \mathbf{p}_i) + \delta(\mathbf{r} - \mathbf{n}_i)] \tag{4.3a}$$

$$\langle \hat{n}(\mathbf{r}) \, \hat{n}(\mathbf{r}') \rangle = \frac{1}{\Xi} \sum_{N=0}^{\infty} \frac{z^{2N}}{(N!)^2} \int_D \prod_{i=1}^{N} \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \, W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \times \sum_{i=1}^{N} [\delta(\mathbf{r} - \mathbf{p}_i) + \delta(\mathbf{r} - \mathbf{n}_i)] \sum_{j=1}^{N} [\delta(\mathbf{r}' - \mathbf{p}_i) + \delta(\mathbf{r}' - \mathbf{n}_i)] \tag{4.3b}$$

etc., one readily gets the important relations

$$\int_D \mathrm{d}^2 r \, n(\mathbf{r}) = z \frac{\partial}{\partial z} \ln \Xi \tag{4.4a}$$

$$\int_D \mathrm{d}^2 r' \, S_n(\mathbf{r}, \mathbf{r}') = z \frac{\partial}{\partial z} n(\mathbf{r}) \tag{4.4b}$$

etc.

The domain of interest in this section is the disk of radius R, $D = \{|\mathbf{r}| \leqslant R\}$. When one rescales the particle coordinates as $\mathbf{p}_i = R\mathbf{p}'_i$ and $\mathbf{n}_i = R\mathbf{n}'_i$, it becomes evident that $\Xi(z, R)$ depends only on the dimensionless combination $z^2 R^{4-\beta}$. As a consequence,

$$R \frac{\partial \Xi}{\partial R} = \left(2 - \frac{\beta}{2}\right) z \frac{\partial \Xi}{\partial z} \tag{4.5}$$

If one accepts the expected value of $c = -1$ in the universal $\ln R$-term of the large-R expansion (1.24) ($\chi = 1$ for the disk), $\ln \Xi = -\beta \Omega$ takes the form

$$\ln \Xi(z, \beta, R) = (\pi R^2) \, \beta p - (2\pi R) \, \beta \gamma - \tfrac{1}{6} \ln(z^{1/(2-\beta/2)} R) + \text{const} + \cdots \tag{4.6}$$

where

$$\beta p = f_V(\beta)\, z^{1/(1-\beta/4)} \tag{4.7}$$

p is the bulk pressure, and

$$\beta\gamma = f_S(\beta)\, z^{1/(2-\beta/2)} \tag{4.8}$$

γ is the surface tension. The bulk particle density n is given by the equation

$$n = z\,\frac{\partial(\beta p)}{\partial z} = \frac{1}{1-(\beta/4)}\,\beta p \tag{4.9}$$

which gives the equation of state (1.11).

The particle density $n(r, R)$ depends on both the radius R and the distance $0 \leqslant r \leqslant R$ from the center of the disk. By using the above mentioned scaling transformation of particle coordinates, one has

$$n(r, R) = z^{1/(1-\beta/4)} g(z^{1/(2-\beta/2)} r,\, z^{1/(2-\beta/2)} R) \tag{4.10}$$

with an unknown function g. The origin can be moved to the boundary via the coordinate transformation $x = R - r$. In order to distinguish between the two different functions, we will use the obvious notation

$$n_R(x) \equiv n(R-x, R) \tag{4.11}$$

Similarly as in (4.10),

$$n_R(x) = z^{1/(1-\beta/4)} \bar{g}(z^{1/(2-\beta/2)} x,\, z^{1/(2-\beta/2)} R) \tag{4.12}$$

with an unknown function $\bar{g}$ different from g. The definition (4.11) implies

$$\frac{\partial n(r, R)}{\partial r} = -\frac{\partial n_R(x)}{\partial x} \tag{4.13a}$$

$$\frac{\partial n(r, R)}{\partial R} = \frac{\partial n_R(x)}{\partial x} + \frac{\partial n_R(x)}{\partial R} \tag{4.13b}$$

The transition from the disk of radius R to a rectilinear hard wall can be understood as the limiting $R \to \infty$ procedure, with $\lim_{R\to\infty} n_R(x) = n(x)$ being the particle density at a finite distance $x \geqslant 0$ from the plain hard wall.

It is evident that the derivative of $\int_0^R \prod_{i=1}^N \mathrm{d}^2 p_i\, \mathrm{d}^2 n_i\, W_\beta^{(N)}(\mathbf{p}, \mathbf{n})$ with respect to R is $2\pi R \int_0^R \prod_{i=1}^N \mathrm{d}^2 p_i\, \mathrm{d}^2 n_i\, W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \sum_{i=1}^N [\delta(\mathbf{p}_i - \mathbf{R}) + \delta(\mathbf{n}_i - \mathbf{R})]$ where $\mathbf{R}$ is a position vector of any point at the disk boundary. Hereinafter, the integration range from 0 to R formally means that $0 \leqslant \{|\mathbf{p}_i|, |\mathbf{n}_i|\} \leqslant R$. As a result,

$$\frac{\partial}{\partial R} \ln \Xi = 2\pi R n_R(0) \tag{4.14}$$

With regard to Eqs. (4.6) and (4.9), the finite-size correction of the particle density at the boundary reads

$$n_R(0) = \left(1 - \frac{\beta}{4}\right) n - \frac{1}{R} (\beta\gamma) - \frac{1}{12\pi R^2} + \cdots \tag{4.15}$$

The well-known contact theorem $n(0) = [1 - (\beta/4)]\, n$[45, 46] results as the $R \to \infty$ limit of Eq. (4.15).

By using the same procedure for $\Xi n(r, R)$ one arrives at

$$\frac{\partial}{\partial R} [\Xi n(r, R)] = \Xi R \int_{-\pi}^{\pi} \mathrm{d}\varphi [\langle \hat{n}(\mathbf{r})\, \hat{n}(R, \varphi) \rangle - n(r)\, \delta(\mathbf{r} - \mathbf{R})] \tag{4.16}$$

where, in polar coordinates, $\mathbf{R} = (R, \varphi)$. Using Eq. (4.13b) and after some simple algebra, one can pass from (4.16) to

$$\frac{\partial n_R(x)}{\partial x} + \frac{\partial n_R(x)}{\partial R} = R \int_{-\pi}^{\pi} \mathrm{d}\varphi\, U_n(\mathbf{r}, \mathbf{R}) \tag{4.17}$$

In the limit $R \to \infty$ and for a fixed x, $n_R(x) = n(x) + O(1/R)$, so $\lim_{R \to \infty} \partial n_R(x)/\partial R \to 0$. When one introduces the y-coordinate as follows $y = R\varphi$, Eq. (4.17) takes the form

$$\frac{\partial n(x)}{\partial x} = \int_{-\infty}^{\infty} \mathrm{d}y\, U_n(0, x; y) \tag{4.18}$$

and we recover the two-dimensional version of the WLMB equation.[47, 48]

We are now ready to take advantage of the exact grand-canonical relation (4.4b). With regard to the scaling form of $n_R(x)$, Eq. (4.12), it holds

$$\left(2 - \frac{\beta}{2}\right) z \frac{\partial n_R(x)}{\partial z} = 2 n_R(x) + x \frac{\partial n_R(x)}{\partial x} + R \frac{\partial n_R(x)}{\partial R} \tag{4.19}$$

In the limit $R\to\infty$, the last term on the rhs of Eq. (4.19) vanishes. Inserting the resulting $z\partial n(x)/\partial z$ into (4.4b), one finally obtains

$$\int_0^\infty \mathrm{d}x' \int_{-\infty}^\infty \mathrm{d}y\, S_n(x,x';y) = \frac{n(x)}{1-(\beta/4)} + \frac{x}{2[1-(\beta/4)]}\frac{\partial n(x)}{\partial x} \tag{4.20}$$

This is the local form of the compressibility sum rule in the presence of a plain hard wall. Indeed, in the limit $x\to\infty$, $n(x)\to n$ and $\partial n(x)/\partial x\to 0$ faster than any power-law due to screening. Consequently, in the bulk, (4.20) reproduces (1.22). It is straightforward to prove that Eq. (4.20) is valid for any value of ϵ_W.

5. MÖBIUS INVARIANCE AND SUM RULES

We now consider the TCP confined to a two-dimensional domain D, and study the action of the Möbius conformal transformation

$$z' = \frac{az+b}{cz+d}, \qquad z = \frac{dz'-b}{-cz'+a} \tag{5.1}$$

(with free complex parameters $ad-bc\neq 0$) of complex particle coordinates $(z,\bar{z})$ on the configuration integral $Q_{N,N}$ (4.1b). Under the conformal transformation (5.1), the domain D is mapped onto the one denoted as D', the surface element $\mathrm{d}z\,\mathrm{d}\bar{z}$ is written as

$$\mathrm{d}z\,\mathrm{d}\bar{z} = \frac{(ad-bc)(\bar{a}\bar{d}-\bar{b}\bar{c})}{(a-cz')^2\,(\bar{a}-\bar{c}\bar{z}')^2}\,\mathrm{d}z'\,\mathrm{d}\bar{z}' \tag{5.2}$$

the square of the distance between two particles takes the form

$$|z_i-z_j|^2 = \frac{(ad-bc)(\bar{a}\bar{d}-\bar{b}\bar{c})}{(a-cz'_i)(\bar{a}-\bar{c}\bar{z}'_i)(a-cz'_j)(\bar{a}-\bar{c}\bar{z}'_j)}\,|z'_i-z'_j|^2 \tag{5.3}$$

so that the Boltzmann factor (4.2) reads

$$W_\beta^{(N)}(\mathbf{p},\mathbf{n}) = [(ad-bc)(\bar{a}\bar{d}-\bar{b}\bar{c})]^{-N\beta/2} \times \prod_{i=1}^N [(a-cp'_i)(\bar{a}-\bar{c}\bar{p}'_i)]^{\beta/2}\,[(a-cn'_i)(\bar{a}-\bar{c}\bar{n}'_i)]^{\beta/2}\,W_\beta^{(N)}(\mathbf{p}',\mathbf{n}') \tag{5.4}$$

Finally, putting $c = \bar{c} = 1$ and $a = r_0$, $\bar{a} = \bar{r}_0$, one finds

$$\int_D \prod_{i=1}^{N} \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \, W_\beta^{(N)}(\mathbf{p}, \mathbf{n})$$
$$= [(b - r_0 d)(\bar{b} - \bar{r}_0 \bar{d})]^{N[2-(\beta/2)]} \int_{D'} \prod_{i=1}^{N} \frac{\mathrm{d}^2 p_i}{|\mathbf{r}_0 - \mathbf{p}_i|^{4-\beta}} \frac{\mathrm{d}^2 n_i}{|\mathbf{r}_0 - \mathbf{n}_i|^{4-\beta}} W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \tag{5.5}$$

This means that the configuration integral of the TCP Coulomb system confined to the domain D is mappable onto the one of the TCP Coulomb system (with the same number of charged particles and at the same inverse temperature β) confined to the new domain D' and being under an additional action of an external gravitational source with specific coupling strength equal to $4-\beta$. Since in the stability range of $\beta < 2$ the gravitational coupling $4-\beta > 2$ and the lhs of Eq. (5.5) is finite, the formalism must ensure that, for any values of free complex parameters b and d such that $(b/d) \neq r_0$, the gravitational source is localized outside of the domain D' in order to prevent the divergence of the rhs of Eq. (5.5) due to the gravitational collapse. According to the definition (4.1a), the relation between the configuration integrals (5.5) also implies an analogous relation between the grand partition functions,

$$\Xi(z, \beta, D) = \Xi(\tilde{z}, \beta, D' \mid \mathbf{r}_0) \tag{5.6a}$$

$$\tilde{z} = z[(b - r_0 d)(\bar{b} - \bar{r}_0 \bar{d})]^{1-(\beta/4)} \tag{5.6b}$$

Taking for the domain a disk with its center at the origin, $D = \{|r| \leqslant R\}$, let us set $b = R^2$ and $d = \bar{r}_0$ $(r_0 \bar{r}_0 \neq R^2)$, besides the already chosen $c = 1$ and $a = r_0$, so that the Möbius conformal transformation (5.1) takes the form

$$z' = \frac{r_0 z + R^2}{z + \bar{r}_0}, \qquad z = \frac{\bar{r}_0 z' - R^2}{-z' + r_0} \tag{5.7}$$

It is straightforward to show that under this transformation the disk $z\bar{z} \leqslant R^2$ maps onto a "dual" domain D' defined by the inequality

$$(R^2 - r_0 \bar{r}_0)(R^2 - z'\bar{z}') \leqslant 0 \tag{5.8}$$

If the gravitational point is outside of the disk, i.e., $r_0 \bar{r}_0 > R^2$, the transformation (5.7) maps the disk onto itself. Relation (5.5) results into

$$\int_0^R \prod_{i=1}^N \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \, W_\beta^{(N)}(\mathbf{p}, \mathbf{n})$$
$$= (r_0 \bar{r}_0 - R^2)^{N(4-\beta)} \int_0^R \prod_{i=1}^N \frac{\mathrm{d}^2 p_i}{|\mathbf{r}_0 - \mathbf{p}_i|^{4-\beta}} \frac{\mathrm{d}^2 n_i}{|\mathbf{r}_0 - \mathbf{n}_i|^{4-\beta}} W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \tag{5.9}$$

If $r_0 \bar{r}_0 < R^2$, the disk is mapped onto its complement in the two-dimensional space and Eq. (5.5) gives

$$\int_0^R \prod_{i=1}^N \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \, W_\beta^{(N)}(\mathbf{p}, \mathbf{n})$$
$$= (R^2 - r_0 \bar{r}_0)^{N(4-\beta)} \int_R^\infty \prod_{i=1}^N \frac{\mathrm{d}^2 p_i}{|\mathbf{r}_0 - \mathbf{p}_i|^{4-\beta}} \frac{\mathrm{d}^2 n_i}{|\mathbf{r}_0 - \mathbf{n}_i|^{4-\beta}} W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \tag{5.10}$$

In both cases, the gravitational source lies outside of the dual domain, as is required by the condition of stability (see the comment in the above paragraph). There exists a special Möbius transformation which maps the disk directly onto a half-plane, but we will not discuss this case.

Although the present paragraph will not be used in the following, let us remark that the self-duality of the disk system (5.9) produces a relation between the grand partition functions of type (5.6), with $D' = D =$ disk and $\tilde{z} = z(r_0 \bar{r}_0 - R^2)^{2-(\beta/2)}$. Since $\Xi(z, \beta, R)$ depends on β and the combination $z^2 R^{4-\beta}$, $\Xi(\tilde{z}, \beta, R|\mathbf{r}_0)$ will depend on β and the combination $\tilde{z}^2 R^{4-\beta}/(r_0 \bar{r}_0 - R^2)^{4-\beta}$. Let $x = |r_0| - R$ be the distance of the gravitational point from the surface of the disk. The consequent ratio $R/(2xR + x^2)$ diverges in the limit $R \to \infty$, $x \to 0$. In this case, the asymptotic formula (4.6) can be applied. When x is finite, the asymptotic formula (4.6) is applicable only when one multiplies z by an appropriate R-dependent constant via changing the zero reference energy of the gravitational potential.

We will now use the self-dual relation (5.9) to derive a new sum rule (5.14) and its more detailed version (5.17) for the density structure function S_n in the case of the disk geometry. First, we rewrite the rhs of (5.9) as follows

$$\int_0^R \prod_{i=1}^N \mathrm{d}^2 p_i \, \mathrm{d}^2 n_i \left[\frac{1 - R^2/(r_0 \bar{r}_0)}{(1 - p_i/r_0)(1 - \bar{p}_i/\bar{r}_0)} \right]^{2-(\beta/2)}$$
$$\times \left[\frac{1 - R^2/(r_0 \bar{r}_0)}{(1 - n_i/r_0)(1 - \bar{n}_i/\bar{r}_0)} \right]^{2-(\beta/2)} W_\beta^{(N)}(\mathbf{p}, \mathbf{n}) \tag{5.11}$$

Then, since $R^2/(r_0 \bar{r}_0) < 1$, $|p_i/r_0| < 1$ and $|n_i/r_0| < 1$, we perform the large-$|r_0|$ expansion

$$\left[\frac{1-R^2/(r_0\bar{r}_0)}{(1-z/r_0)(1-\bar{z}/\bar{r}_0)}\right]^{2-(\beta/2)}$$
$$=1+\left(2-\frac{\beta}{2}\right)\frac{z}{r_0}+\left(2-\frac{\beta}{2}\right)\frac{\bar{z}}{\bar{r}_0}-\left(2-\frac{\beta}{2}\right)\frac{R^2}{r_0\bar{r}_0}+\left(2-\frac{\beta}{2}\right)^2\frac{z\bar{z}}{r_0\bar{r}_0}+\cdots \tag{5.12}$$

for each $z=\{p_i, n_i\}$. The term of order $(r_0\bar{r}_0)^0$ in (5.11) exactly reproduces the lhs of (5.9), and the coefficients of higher-order terms in inverse powers of r_0 and $\bar{r}_0$ must be identically equal to zero. The terms $1/r_0$ and $1/\bar{r}_0$ trivially vanish. The $1/(r_0\bar{r}_0)$ term vanishes when

$$\int_0^R \prod_{i=1}^N d^2p_i\, d^2n_i \sum_{j,k=1}^N (p_j+n_j)(\bar{p}_k+\bar{n}_k)\, W_\beta^{(N)}(\mathbf{p},\mathbf{n})$$
$$=\frac{2NR^2}{2-(\beta/2)}\int_0^R \prod_{i=1}^N d^2p_i\, d^2n_i\, W_\beta^{(N)}(\mathbf{p},\mathbf{n}) \tag{5.13}$$

Within the grand-canonical formalism (4.1)–(4.3), the equality (5.13) is equivalent to

$$\int_0^R d^2r \int_0^R d^2r'\, \mathbf{r}\cdot\mathbf{r}' S_n^{(R)}(\mathbf{r},\mathbf{r}')=\frac{R^2}{2-(\beta/2)}\, z\frac{\partial}{\partial z}\ln \Xi(z,\beta,R) \tag{5.14}$$

Everything that has been done in the case of the configuration integral can be adapted to the particle density. Let us "rotate" the Möbius transformation (5.7) around the origin by the multiplication factor $(\bar{r}_0/r_0)$ in order to obtain the identity $z'=z$ in the limit $|r_0|\to\infty$, and denote by a the ratio R/r_0 whose absolute value is smaller than 1,

$$z'=\frac{z+aR}{1+(\bar{a}z/R)} \tag{5.15}$$

For a given point $\mathbf{r}=(r,\bar{r})$ in the interior of the disk, the previously developed formalism implies

$$\int_0^R \prod_{i=1}^N d^2p_i\, d^2n_i \sum_{j=1}^N [\delta(\mathbf{r}-\mathbf{p}_j)+\delta(\mathbf{r}-\mathbf{n}_j)]\, W_\beta^{(N)}(\mathbf{p},\mathbf{n})$$
$$=\frac{(1-a\bar{a})^2}{(1+\bar{a}r/R)^2\,(1+a\bar{r}/R)^2}\int_0^R \prod_{i=1}^N d^2p_i\left[\frac{1-a\bar{a}}{(1-\bar{a}p_i/R)(1-a\bar{p}_i/R)}\right]^{2-(\beta/2)} d^2n_i$$
$$\times\left[\frac{1-a\bar{a}}{(1-\bar{a}n_i/R)(1-a\bar{n}_i/R)}\right]^{2-(\beta/2)}\sum_{j=1}^N [\delta(\mathbf{r}'-\mathbf{p}_j)+\delta(\mathbf{r}'-\mathbf{n}_j)]\, W_\beta^{(N)}(\mathbf{p},\mathbf{n}) \tag{5.16}$$

When one expands the rhs of (5.16) in powers of $a = a_x + \mathrm{i}a_y$, $|a| < 1$, the requirement of the nullity of the coefficients attached to a_x and a_y leads to the relation

$$(4-\beta) \int_0^R \mathrm{d}^2r' \, \mathbf{r} \cdot \mathbf{r}' S_n^{(R)}(\mathbf{r}, \mathbf{r}') = 4r^2 n(r, R) + (r^2 - R^2) \, r \frac{\partial}{\partial r} n(r, R) \qquad (5.17)$$

The new sum rule (5.14) results from the more detailed Eq. (5.17) when one integrates $\int_0^R \mathrm{d}^2r$ both sides of the latter, then uses an integration by parts and finally applies the equality (4.4a).

This new sum rule (5.17) has a consequence for the density near the boundary. Let us divide Eq. (5.17) by $|\mathbf{r}|$, and move the origin to the boundary via the coordinate transformation $r = R - x$ described in Section 4. Then, using Eq. (4.19), and in the $R \to \infty$ limit, we find after some simple algebra that

$$n_R(x) = n(x) - \frac{1}{2R} \left\{ (4-\beta) \int_0^\infty \mathrm{d}x' \, x' \int_{-\infty}^\infty \mathrm{d}y \, S_n(x, x'; y) \right.$$
$$\left. - 4xn(x) - x^2 \frac{\partial n(x)}{\partial x} \right\} + \cdots \qquad (5.18)$$

This formula determines the leading correction of the density $n_R(x)$, at a fixed distance x from the wall, with respect to its asymptotic $R \to \infty$ value $n(x)$ due to the curvature of the confining disk domain. The quantities on the rhs of (5.18) are the ones evaluated for the rectilinear hard wall.

The asymptotic formula (5.18) can be explicitly checked at the boundary, where it gives

$$n_R(0) = n(0) - \frac{4-\beta}{2R} \int_0^\infty \mathrm{d}x \, x \int_{-\infty}^\infty \mathrm{d}y \, S_n(0, x; y) + \cdots \qquad (5.19)$$

Multiplying Eq. (4.18) by x, integrating then over x from 0 to ∞ and performing an integration by parts, one finds

$$-\int_0^\infty \mathrm{d}x [n(x) - n] = \int_0^\infty \mathrm{d}x \, x \int_{-\infty}^\infty \mathrm{d}y [S_n(0, x; y) - n(x) \, \delta(x) \, \delta(y)] \qquad (5.20)$$

On the other hand, according to the relation (4.4a), it holds for the disk

$$\int_0^R \mathrm{d}^2r [n(r, R) - n] = z \frac{\partial \ln \Xi}{\partial z} - n\pi R^2 \qquad (5.21)$$

Assuming the large-R expansion (4.6) and passing to the boundary-distance variable $x=R-r$, Eq. (5.21) takes the form

$$2\pi \int_0^R \mathrm{d}x(R-x)[n_R(x)-n] = -(2\pi R)\, z\, \frac{\partial(\beta\gamma)}{\partial z} - \frac{1}{12(1-(\beta/4))} + \cdots \tag{5.22}$$

In the limit $R\to\infty$, this equation implies the obvious boundary relation

$$\int_0^\infty \mathrm{d}x[n(x)-n] = -z\, \frac{\partial(\beta\gamma)}{\partial z} \tag{5.23}$$

Taking into account the z-dependence of $\beta\gamma$ (4.8) in (5.23) and considering the previously derived relations (5.19) and (5.20), large-R asymptotics (4.15) is reproduced correctly up to the $(1/R)$-term, which confirms the validity of (5.19).

The asymptotic formula (5.18) implies a generalization of the second moment density sum rule (1.23) to the case of a rectilinear wall. Indeed, the local version of the compressibility sum rule (4.20) can be used for rewriting (5.18) as

$$n_R(x) = n(x) - \frac{1}{2R}\left\{(4-\beta)\int_0^\infty \mathrm{d}x'(x'-x)\int_{-\infty}^\infty \mathrm{d}y\, S_n(x,x';y) + x^2\frac{\partial n(x)}{\partial x}\right\} + \cdots \tag{5.24}$$

We will suppose that the half-infinite Coulomb system has good screening properties into the bulk, i.e., $[n(x)-n]$ decays faster than any inverse-power law as $x\to\infty$ and all moments $\int_0^\infty \mathrm{d}x\, x^i[n(x)-n]$ exist. From Eqs. (5.22) and (5.23), one then gets in the limit of large R

$$2\pi R\int_0^\infty \mathrm{d}x[n_R(x)-n(x)] - 2\pi\int_0^\infty \mathrm{d}x\, x[n(x)-n] = -\frac{1}{12(1-(\beta/4))} \tag{5.25}$$

Inserting (5.24) into (5.25) and after some algebra we arrive at

$$\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}x'(x'-x)\int_{-\infty}^\infty \mathrm{d}y\, S_n(x,x';y) = \frac{1}{48\pi(1-(\beta/4))^2} \tag{5.26}$$

Actually, it can be shown that the boundary sum rule (5.26) can be derived directly from the second-moment sum rule in the bulk (1.23) by using a simple assumption. Indeed, $S_n(x, x'; y)$ can be decomposed as the sum of the bulk structure factor plus a surface term:

$$S_n(x, x'; y) = S_n^{\text{bulk}}(r) + S_n^{\text{surface}}(x, x'; y) \tag{5.27}$$

where $r = [(x-x')^2 + y^2]^{1/2}$ and $S_n^{\text{bulk}}(r)$ is the bulk density structure factor appearing in (1.23) [(5.27) is just a definition of S_n^{surface}]. When this decomposition is used in the lhs of (5.26), assuming that $S_n^{\text{surface}}(x, x'; y)$ has a fast decay when x or x' or both go to infinity (see for instance the explicit expressions in the weak-coupling or $\beta = 2$ cases considered in Sections 3.1 and 3.2), the corresponding integral is absolutely convergent, the order of the integrations on x and x' can be freely interchanged, and since $(x-x')\, S_n^{\text{surface}}(x, x'; y)$ is odd under the interchange of x and x', this integral vanishes. This reasoning however does not apply to the contribution of the bulk structure factor which does not decay to 0 when both x and x' go to infinity for a fixed value of $x-x'$; the corresponding integral is not absolutely convergent, and the order of the integrations cannot be changed. Thus, only the bulk part of S_n contributes to the lhs I of (5.26) which can be rewritten as

$$I = \int_0^\infty \mathrm{d}x \int_{-x}^\infty \mathrm{d}s\, s \int_{-\infty}^\infty \mathrm{d}y\, S_n^{\text{bulk}}(|s|; y) \tag{5.28}$$

where we have used the integration variable $s = x' - x$ rather than x' and made explicit that S_n^{bulk} depends on x and x' only through $|s|$. Performing the integration on x by parts gives

$$I = x \int_{-x}^\infty \mathrm{d}s\, s \int_{-\infty}^\infty \mathrm{d}y\, S_n^{\text{bulk}}(|s|; y)\Big|_{x=0}^{x=\infty} + \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}y\, x^2 S_n^{\text{bulk}}(|x|; y) \tag{5.29}$$

Because S_n^{bulk} has a fast (exponential) decay at infinity, the first term in the rhs of (5.29) vanishes. Since $S_n^{\text{bulk}}(|x|; y)$ is a function of $r = (x^2 + y^2)^{1/2}$ only, I is $1/4$ of the lhs of (1.23), and (1.23) results into (5.26). It should be noted that the above derivation of the boundary sum rule (5.26) still holds for an arbitrary value of the wall dielectric constant ϵ_W, including the special cases $\epsilon_W = 0$ and $\epsilon_W = \infty$.

6. UNIVERSAL FINITE-SIZE CORRECTION FOR THE DISK

In most considerations of the previous two sections, we have assumed the validity of the large-R expansion of $\ln \Xi$ (4.6) for the disk, in particular the value $-1/6$ of the coefficient of the universal $\ln R$-term. Here, we prove this assumption.

Using the formula $|\mathbf{r}-\mathbf{r}'|^2=|\mathbf{r}|^2+|\mathbf{r}'|^2-2\mathbf{r}\cdot\mathbf{r}'$, we write

$$\int_0^R d^2r \int_0^R d^2r'\,|\mathbf{r}-\mathbf{r}'|^2\,S_n^{(R)}(\mathbf{r},\mathbf{r}') = 2z\frac{\partial}{\partial z}\left[\int_0^R d^2r\,|\mathbf{r}|^2\,n(r,R)-\frac{R^2}{2-(\beta/2)}\ln\Xi\right] \tag{6.1}$$

where we have used the relation (4.4b) and the new sum rule (5.14). Let us consider the large-R expansion (4.6) in a general form

$$\ln\Xi=(\pi R^2)\left(1-\frac{\beta}{4}\right)n-(2\pi R)\,\beta\gamma+f \tag{6.2}$$

where $n\propto z^{1/(1-\beta/4)}$, $\beta\gamma\propto z^{1/(2-\beta/2)}$ and f is an as-yet-undetermined function of β and the combination $z^2R^{4-\beta}$. The representation (6.2) when combined with (5.21) implies

$$\beta\gamma=\frac{1-(\beta/4)}{\pi R}\left\{z\frac{\partial f}{\partial z}-\int_0^R d^2r[n(r,R)-n]\right\} \tag{6.3}$$

Substituting $\beta\gamma$ in (6.2) by the expression (6.3) and then inserting $\ln\Xi$ into (6.1), one obtains

$$\frac{1}{\pi R^2}\int_0^R d^2r\int_0^R d^2r'\,|\mathbf{r}-\mathbf{r}'|^2\,S_n^{(R)}(\mathbf{r},\mathbf{r}') = -\frac{2}{\pi}z\frac{\partial}{\partial z}\int_0^R d^2r\left(1-\frac{r^2}{R^2}\right)[n(r,R)-n]-\frac{1}{\pi(1-(\beta/4))}z\frac{\partial}{\partial z}f+\frac{2}{\pi}\left(z\frac{\partial}{\partial z}\right)^2 f \tag{6.4}$$

This equation, which is exact for an arbitrary disk radius R, will now be analyzed in the limit $R\to\infty$. In that limit, since $S_n^{(R)}(\mathbf{r},\mathbf{r}')$ decays along the boundary like $1/|\mathbf{r}-\mathbf{r}'|^4$ (see Section 3), there is no relevant surface contribution to the integral on the lhs of (6.4), and we can make use of the bulk sum rule (1.23) derived in ref. 25,

$$\lim_{R\to\infty}\frac{1}{\pi R^2}\int_0^R d^2r\int_0^R d^2r'\,|\mathbf{r}-\mathbf{r}'|^2\,S_n^{(R)}(\mathbf{r},\mathbf{r}')=\frac{1}{12\pi(1-(\beta/4))^2} \tag{6.5}$$

As concerns the rhs of (6.4), the unknown integral over the disk area can be represented in the $x=R-r$ coordinate as

$$\int_0^R \mathrm{d}^2 r \left(1-\frac{r^2}{R^2}\right)[n(r,R)-n] = 4\pi \int_0^R \mathrm{d}x\, x \left(1-\frac{x}{R}\right)\left(1+\frac{x}{2R}\right)[n_R(x)-n] \tag{6.6}$$

In the limit $R\to\infty$, due to good screening properties of the plasma, only the term $\propto \int_0^\infty \mathrm{d}x\, x[n(x)-n]$ survives. Based on the scaling analysis of Section 3, since $n(x)=n\bar{g}(\sqrt{n}\,x)$, this integral is dimensionless, and therefore its derivative with respect to z vanishes. Consequently, without the integral on the rhs of Eq. (6.4), the solution of that equation combined with (6.5) reads

$$f = -\frac{1}{6}\ln(Rz^{1/(2-\beta/2)}) + \mathrm{const} + \cdots \tag{6.7}$$

in full agreement with the expected universal large-R behavior (4.6).

7. CONCLUDING REMARKS

The peculiarities of the charge correlation in a Coulomb fluid are determined by the screening effect caused by the long-ranged tail of the Coulomb potential. The density correlation function in a charge-symmetrical two-component plasma, decoupled from the charge one at the level of the OZ equation [see relations (2.1)], does not feel this tail directly. Yet, the density correlation bears some similarity with the charge correlation. Although the density correlation decays always faster than the charge one, it still exhibits an algebraically slow decay $\sim |\mathbf{y}|^{-2\nu}$ (3.4) at large distances $|\mathbf{y}|$ along a rectilinear hard wall in any dimension ν. An analogous phenomenon for the charge correlation, which decays like $\sim |\mathbf{y}|^{-\nu}$ (3.3), is related to an asymmetry of the screening cloud near the wall.

In this paper, we have concentrated mainly on the two-dimensional TCP of pointlike charges. The form-factor analysis of the equivalent sine-Gordon model was done in the bulk case. In contrast to the charge correlation, the density one exhibits a discontinuity of the slope of the correlation length, namely at point $\beta=1$ [see relations (2.31) and (2.32)]. At the collapse point $\beta=2$, as seen in Eqs. (2.10) and (2.11), the charge and density correlation lengths become the same and the difference in the asymptotic decay is only in the inverse-power law prefactors.

The TCP in contact with a rectilinear hard wall of dielectric constant ϵ_W was mapped onto an integrable boundary sine-Gordon model in two cases: $\epsilon_W \to \infty$ (ideal metal wall, Dirichlet boundary condition)[5] and $\epsilon_W = 0$ (ideal dielectric wall, Neumann boundary condition).[6] Interestingly, just in these two cases the charge and density correlation functions decay exponentially also along the boundary. The next potentially integrable case corresponds to the plain hard wall with $\epsilon_W = 1$: at least, the explicit solution is available at the free-fermion point.[9] For this case, we have suggested an explicit form of the asymptotic characteristics along the wall, $f_\rho(x, x')$ (3.15) and $f_n(x, x')$ (3.16), as a natural interpolation between the exact results in the $\beta \to 0$ limit and at $\beta = 2$. The universal value of the prefactor $= 1/(2\pi^2)$ in f_n deserves attention. To check the validity of our conjectures, one should go beyond the Debye–Hückel limit.

The scaling analysis of Section 4 leads to a new local version of the compressibility sum rule in the presence of a hard wall (4.20).

In the crucial Section 5, a symmetry of the TCP confined to a disk with respect to the Möbius conformal transformation of particle coordinates, which induces a gravitational source point interacting with the particles, is established. This symmetry is of interest by itself: it produces a path between the pure Coulomb system and the one under the action of a gravitational point. The gravitational point acts in the same way on positive and negative charges, so that the total particle density is relevant. As a result, the new sum rule (5.14) for the density correlation function and its local version (5.17) are derived for the disk geometry of the Coulomb system. The finite-size formula (5.18) for the particle density at a fixed distance from the wall and the boundary version (5.26) of the bulk second moment of S_n (1.23) also deserve attention.

The new density sum rule (5.14) is used in Section 6 to derive the universal finite-size correction term of the grand potential for the disk geometry. We would like to stress a fundamental difference between the derivation of the universal finite-size correction term for a system which is finite in each direction (a disk or a sphere, a *density* sum rule is relevant) and for a system which is infinite at least in one direction (a strip or a ν-dimensional slab, a *charge* sum rule is relevant[33, 34]).

In most cases, we have restricted ourselves to the symmetrical-charge TCP and $\epsilon_W = 1$ in order to maintain the clarity of presentation. The extension of the formalism to the asymmetric TCP and arbitrary ϵ_W is usually straightforward.

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